ELSEVIER

Topology and its Applications 74 (1996) 97–107

TOPOLOGY AND ITS APPLICATIONS

Strongly normal ideals on $\mathcal{P}_\kappa\lambda$ and the Sup-function

Yoshihiro Abe [1]

Department of Mathematics, Kanagawa University, Yokohama, Japan

Received 1 May 1995; revised 9 November 1995

Abstract

We shall prove that for any regular λ and strongly normal λ-saturated ideal I on $\mathcal{P}_\kappa\lambda$ the Sup-function is one-to-one on some $X \in I^*$, generalizing Solovay's theorem for normal ultrafilters.

Keywords: Strong normality; Saturated ideals; $\mathcal{P}_\kappa\lambda$; Generic ultrapowers; Rudin–Keisler ordering

AMS classification: 03E; 04

0. Introduction

Solovay proved the following well-known theorem for the Sup-function in [15].

Solovay's theorem. *Suppose that λ is regular and U is a normal ultrafilter on $\mathcal{P}_\kappa\lambda$. If $f:\mathcal{P}_\kappa\lambda \to \lambda$ is defined by $f(x) = \sup(x)$, then $f|X$ is one-to-one on some $X \in U$.*

Johnson [11] extended it for the non λ-Shelah ideal, $NSh_{\kappa\lambda}$ for $\lambda = \mu^+$. The argument is available for the non almost λ-ineffable ideal with any regular λ. These ideals can be seen as a weakening of normal ultrafilters in terms of the partition property.

If we turn our attention to the fact in [1] that $\lambda^{<\kappa} = \lambda$ if $\mathcal{P}_\kappa\lambda$ carries a normal λ-saturated ideal and $2^{<\kappa} \leqslant \lambda$, it may be natural to ask;

Does Solovay's theorem also hold for normal λ-saturated ideals on $\mathcal{P}_\kappa\lambda$?

In the first section definitions and basic facts for ideals on $\mathcal{P}_\kappa\lambda$ are stated. In the second section we extend Solovay's theorem to strongly normal λ-saturated ideals on

[1] This research was partially supported by Grant-in-Aid for Scientific Research (No. 06640178 and No. 06302014), Ministry of Education, Science and Culture of Japan. The author is grateful to the referee for pointing out many errors. E-mail: yabe@cc.kanagawa-u.ac.jp.

SSDI 0166-8641(96)00048-X

$\mathcal{P}_\kappa\lambda$ with λ regular. In the last section we introduce the Rudin–Keisler ordering to the ideals on $\mathcal{P}_\kappa\lambda$. Minimal ideals in this ordering are considered.

Throughout this paper κ is a regular uncountable cardinal and λ a cardinal above κ. For such κ and λ, $\mathcal{P}_\kappa\lambda = \{x \subset \lambda\colon |x| < \kappa\}$.

1. Basic facts on ideals

In this section we review the basic definitions and the facts for ideals on $\mathcal{P}_\kappa\lambda$ used in the later sections.

Definition. I is an *ideal* on $\mathcal{P}_\kappa\lambda$ if $I \subset \mathcal{P}(\mathcal{P}_\kappa\lambda)$ and

(i) $\emptyset \in I$ and $\mathcal{P}_\kappa\lambda \notin I$ (I is proper),

(ii) if $X \in I$ and $Y \subset X$, then $Y \in I$,

(iii) if $X \in I$ and $Y \in I$, then $X \cup Y \in I$.

I is *κ-complete* if I is closed under union of less than κ many members.

I is *fine* if for all $\alpha < \lambda$, $\{x \in \mathcal{P}_\kappa\lambda\colon \alpha \notin x\} \in I$. For the sake of convenience, all ideals are assumed to be κ-complete and fine.

I is *normal* if for any $\{X_\alpha\colon \alpha < \lambda\} \subset I$, $\nabla_{\alpha<\lambda} X_\alpha = \{x\colon x \in X_\alpha$ for some $\alpha \in x\} \in I$. $\nabla_{\alpha<\lambda} X_\alpha$ is called the *diagonal union* of $\{X_\alpha\colon \alpha < \lambda\}$.

I is *strongly normal* if for any $\{X_a\colon a \in \mathcal{P}_\kappa\lambda\} \subset I$, $\nabla_< X_a = \{x\colon x \in X_a$ for some $a \subset x$ with $|a| < |x \cap \kappa|\} \in I$. So, every strongly normal ideal is normal.

$I^+ = \{X \subset \mathcal{P}_\kappa\lambda\colon X \notin I\}$ and $I^* = \{X \subset \mathcal{P}_\kappa\lambda\colon \mathcal{P}_\kappa\lambda - X \in I\}$. Any $X \in I^+$ is called *I-positive*.

In [10], Jech generalized the notion of closed unbounded sets to $\mathcal{P}_\kappa\lambda$.

Definition. $X \subset \mathcal{P}_\kappa\lambda$ is *unbounded* if for any $a \in \mathcal{P}_\kappa\lambda$, we have $x \in X$ such that $a \subset x$.

X is *closed* if for any $\subset$-increasing sequence $\{x_\alpha\colon \alpha < \delta < \kappa\} \subset X$, $\bigcup_{\alpha<\delta} x_\alpha \in X$.

X is *stationary* if $X \cap C \neq \emptyset$ for all closed unbounded sets C.

$I_{\kappa\lambda} = \{X \subset \mathcal{P}_\kappa\lambda\colon X$ is not unbounded$\}$ which is the smallest ideal on $\mathcal{P}_\kappa\lambda$.

$NS_{\kappa\lambda} = \{X \subset \mathcal{P}_\kappa\lambda\colon X$ is not stationary$\}$.

Theorem 1.1 [6,7,10]. (1) *For any $X \in I^+_{\kappa\lambda}$, $|X| \geqslant \lambda$.*

(2) *$NS_{\kappa\lambda}$ is the smallest normal ideal on $\mathcal{P}_\kappa\lambda$.*

(3) *Strongly normal ideals on $\mathcal{P}_\kappa\lambda$ exist iff κ is Mahlo or $\kappa = \nu^+$ with $\nu^{<\nu} = \nu$.*

(4) *Every normal ultrafilter on $\mathcal{P}_\kappa\lambda$ is strongly normal.*

Definition. $X, Y \subset \mathcal{P}_\kappa\lambda$ are *almost disjoint* with respect to I if $X \cap Y \in I$. I is *η-saturated* if there is no pairwise almost disjoint family of η many I-positive sets.

It was shown in [1] that $\lambda^{<\kappa}$ is small under the existence of saturated ideals.

Theorem 1.2. (1) *If there is a normal λ-saturated ideal on $\mathcal{P}_\kappa\lambda$, then*

$$\lambda^{<\kappa} = \begin{cases} 2^{<\kappa} \cdot \lambda & \text{if } \mathrm{cf}(\lambda) \geqslant \kappa, \\ 2^{<\kappa} \cdot \lambda^+ & \text{if } \mathrm{cf}(\lambda) < \kappa. \end{cases}$$

(2) *If there is a normal λ^+-saturated ideal on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} \leqslant 2^{<\kappa} \cdot \lambda^+$.*

Strong normality is related to following notion of distributivity.

Definition. For $A \in I^+$, an *I-partition of A* is a maximal almost disjoint family $\subset \mathcal{P}(A) \cap I^+$. I is *(μ, ν)-distributive* if whenever $A \in I^+$ and $\langle W_\alpha \mid \alpha < \mu \rangle$ is a sequence of I-partitions of A with $|W_\alpha| \leqslant \nu$ for all $\alpha < \mu$, there is a $B \in \mathcal{P}(A) \cap I^+$ and a sequence $\langle X_\alpha \mid \alpha < \mu \rangle$ such that $X_\alpha \in W_\alpha$ and $B - X_\alpha \in I$ for all $\alpha < \mu$. Such a sequence $\langle X_\alpha \mid \alpha < \mu \rangle$ is called a *branch*.

The following is due to Johnson [11] and we use it in 2.7.

Lemma 1.3. *For any normal ideal I and $\mu < \kappa$, I is (μ, λ)-distributive iff for any $X \in I^+$ and $f : X \to {}^\mu\lambda$ with $f(x) \in {}^\mu x$ for all $x \in X$, there is a $Y \in \mathcal{P}(X) \cap I^+$ such that $f|Y$ is constant.*

Thus, for κ inaccessible, I is strongly normal iff it is normal and (μ, λ)-distributive for all $\mu < \kappa$.

We also use generic ultrapowers. Foreman [9] will be a good reference for example. Consider a poset $\langle P_I, \leqslant \rangle$ in the ground model V and a generic filter G on P_I where

$$P_I = I^+ \text{ and } X \leqslant Y \quad \text{iff} \quad X \subset Y.$$

G is a V-κ-complete V-ultrafilter on $\mathcal{P}_\kappa\lambda$ and we can form, in $V[G]$, an ultrapower of V, $\mathrm{Ult}(V, G) = \{[f]\colon f \in V \text{ is a function with domain } \mathcal{P}_\kappa\lambda\}$ where $[f]$ denotes the equivalence class of functions represented by f defined below.

$$f \equiv_G g \quad \text{iff} \quad \{x \in \mathcal{P}_\kappa\lambda\colon f(x) = g(x)\} \in G,$$
$$f \in_G g \quad \text{iff} \quad \{x \in \mathcal{P}_\kappa\lambda\colon f(x) \in g(x)\} \in G.$$

Then *the fundamental theorem* holds for any formula φ in the language of set theory.

$$\mathrm{Ult}(V, G) \models \varphi([f]) \quad \text{iff} \quad \{x \in \mathcal{P}_\kappa\lambda\colon V \models \varphi(f(x))\} \in G.$$

$j : V \to \mathrm{Ult}(V, G)$ is the canonical generic elementary embedding defined by $j(x) = [c_x]$ where c_x is the constant function with value x.

Definition. I is *prepicitous* if $\mathrm{Ult}(V, G)$ is well-founded.

We write M_G to denote the transitive collapse of $\mathrm{Ult}(V, G)$ when it is well-founded.

Lemma 1.4. (1) *If I is normal, then $[id] = j''\lambda$.*

(2) *If I is normal and λ^+-saturated, then I is prepicitous.*

2. Sup-function

In this section we generalize Solovay's theorem to strongly normal λ-saturated ideals.

Definition. *Sup* denotes the function $f:\mathcal{P}_\kappa\lambda \to \lambda+1$ such that $f(x)=\sup(x)$ for each $x \in \mathcal{P}_\kappa\lambda$.

The main theorem in this section is the following.

Theorem 2.1. *If λ is regular and I is a normal λ-saturated (ω,λ)-distributive ideal on $\mathcal{P}_\kappa\lambda$, then $Sup|X$ is one-to-one for some $X \in I^*$.*

We prove it by series of lemmas. The concept of *weakly ω-Jónsson function* is the key of the proof.

Lemma 2.2. *If λ is regular and I is a normal λ-saturated ideal on $\mathcal{P}_\kappa\lambda$, then*

$$\{x \in \mathcal{P}_\kappa\lambda\colon\ \overline{x} \text{ is regular}\} \in I^*.$$

Proof. I is prepicitous and $[\langle \overline{x} \mid x \in \mathcal{P}_\kappa\lambda\rangle] = \lambda$ by 1.4. Since I is λ-saturated, λ is regular in $V[G]$ hence in M_G. Now the fundamental theorem derives the conclusion. $\square$

The difficulty in extending Solovay's theorem to normal λ-saturated ideals is that $j''\lambda$ is not always closed under ω-sequences. To accomplish this we use distributivity.

Lemma 2.3. *If I is normal (ω,λ)-distributive, then ${}^\omega j''\lambda \cap \mathrm{Ult}(V,G) \subset V$.*

Proof. Suppose that $X \Vdash [f]:\omega \to j''\lambda$, $X \in I^+$ and $f:\mathcal{P}_\kappa\lambda \to V$. We may assume that $f(x):\omega \to x$ for all $x \in X$. Thus, $f(x)(n) \in x$ for any $n \in \omega$ and $x \in X$. Let $Y_n^\gamma = \{x \in X\colon\ f(x)(n) = \gamma\}$ for each $n \in \omega$ and $\gamma < \lambda$. For each n, $Z_n = \{Y_n^\gamma\colon\ \gamma < \lambda\} \cap I^+$ is a disjoint I-partition of X. Now choose any decreasing sequence of I-partitions of X, $\langle W_n \mid n \in \omega\rangle$ such that W_n is a refinement of Z_n. Since I is (ω,λ)-distributive, there is a branch $\langle X_n \mid n \in \omega\rangle$ such that $S = \bigcap\{X_n\colon\ n \in \omega\} \in I^+$. Let $g:\omega \to \lambda$ so that $X_n \subset Y_n^{g(n)}$. It is clear that $S \Vdash [f] = g$. $\square$

Lemma 2.4. *If I is normal (ω,λ)-distributive, then $j''\lambda$ is ω_1-closed and ${}^\omega j''\lambda \in \mathrm{Ult}(V,G)$.*

Proof. Suppose that $f:\omega \to j''\lambda$, $f \in V[G]$. By the previous lemma, $f \in V$. For each $n \in \omega$, there is an $\alpha < \lambda$ such that $f(n) = j(\alpha)$. If we define $g:\omega \to \lambda$ by $g(n) = \alpha$ for each n, $j(g) = f \in \mathrm{Ult}(V,G)$.

$\sup(f''\omega) = \bigcup\{j(g)(n)\colon\ n \in \omega\} = j(\bigcup\{g(n)\colon\ n \in \omega\}) = j(\sup(g''\omega))$. Since λ is regular, $\sup(g''\omega) \in \lambda$. Hence $\sup(f''\omega) \in j''\lambda$. $\square$

The following is a modification of ω-Jónsson functions for our purpose.

Definition. $F: {}^\omega\lambda \to \lambda$ is *weakly ω-Jónsson* over λ if $F''\,{}^\omega x = \lambda$ for any ω_1-closed $x \subset \lambda$ with $|x| = \lambda$.

Lemma 2.5. *If λ is regular and I is a normal λ-saturated, (ω, λ)-distributive ideal on $\mathcal{P}_\kappa\lambda$, then $\{x\colon F|^\omega x$ is weakly ω-Jónsson$\} \in I^*$ for any weakly ω-Jónsson function F over λ.*

Proof. Since I is prepicitous, let $j: V \to M_G \cong \mathrm{Ult}(V, G)$, M_G transitive. Suppose that $M_G \models X \subset j''\lambda \wedge X$ is ω_1-closed $\wedge |X| = |j''\lambda|$.

Then $Y = j^{-1}(X) \in \mathcal{P}(\lambda) \cap V[G]$ and $|Y|^{V[G]} = |Y|^V = \lambda$ since I is λ-saturated.

We first show that Y is ω_1-closed in $V[G]$. Let $S = \{\alpha_n\colon n \in \omega\} \subset Y$ increasing. Then $T = \{j(\alpha_n)\colon n \in \omega\} \subset X$ is also increasing and $T \in M_G$ since $X \subset j''\lambda$ and ${}^\omega j''\lambda \in M_G$. Hence $\alpha = \sup(T) \in X$ because X is ω_1-closed. Now, $j^{-1}(\alpha) \in Y$ and $j^{-1}(\alpha) = \sup(S)$.

Second we show that there is a ω_1-closed $Z \in \mathcal{P}(Y) \cap V$ with $|Z| = \lambda$. Let $\mathbf{1} \Vdash \underline{Y}$ is ω_1-closed cofinal subset of λ. $Z = \{\alpha < \lambda\colon \mathbf{1} \Vdash \check{\alpha} \in \underline{Y}\}$ is obviously ω_1-closed in V. To prove Z is cofinal in λ, let $\beta < \lambda$. We construct a ω-sequence $\{\beta_n\colon n \in \omega\}$ with $\beta = \beta_0$ by induction. Assume that β_n is already defined. Let $A_n \subset \{p \in P_I\colon p \Vdash \beta_n \leqslant \gamma_p \in \underline{Y}$ for some $\gamma_p < \lambda\}$ be any maximal incompatible set and $\beta_{n+1} = \sup\{\gamma_p\colon p \in A_n\}$. Then, $\beta_{n+1} < \lambda$ since $|A_n| < \lambda$. We also have $\mathbf{1} \Vdash \underline{Y} \cap [\beta_n, \beta_{n+1}] \neq \emptyset$. Hence $\beta \leqslant \alpha = \sup\{\beta_n\colon n \in \omega\} < \lambda$ and $\alpha \in Z$ since $\underline{Y}$ is ω_1-closed with the value $\mathbf{1}$. (If $\beta_n \in Z$ for some $n \in \omega$, we stop the induction and set $\beta_n = \alpha$.)

Since F is weakly ω-Jónsson over λ, $F''\,{}^\omega Z = \lambda$. So, for any $\alpha < \lambda$ there exists an $x \in {}^\omega Z$ such that $\alpha = F(x)$. $j(\alpha) = j(F)(j(x)) = j(F)(j''x)$ and $j''x \in {}^\omega j''Y$ because ${}^\omega Z \subset {}^\omega Y$. Hence $j''\lambda = j(F)''({}^\omega j''Y) = j(F)''({}^\omega X)$.

We have proved that $M_G \models j(F)$ is weakly ω-Jónsson over $j''\lambda$. The fundamental theorem and the density argument work. $\square$

Proof of Theorem 2.1. Let F be weakly ω-Jónsson over λ. By Lemmas 2.1, 2.4 and 2.5,

$$X = \{x \in \mathcal{P}_\kappa\lambda\colon F|^\omega x \text{ is weakly } \omega\text{-Jónsson over } x,\ \overline{x} \text{ is regular} \geqslant \omega_1,\\ x \text{ is } \omega_1\text{-closed}\} \in I^*.$$

If $\{x, y\} \subset X$ and $\sup(x) = \sup(y) = \gamma$, then $x \cap y$ is cofinal in γ. Hence $|x| = |x \cap y| = |y|$ which implies $x = F''\,{}^\omega(x \cap y) = y$. $\square$

Next we give some variations of Theorem 2.1 for ideals defined by familiar notions.

Lemma 2.6. *If I is normal and η^+-saturated with $\eta^\omega < \kappa$, then I is (ω, λ)-distributive.*

Proof. Let $X \in I^+$, $f: X \to {}^\omega\lambda$ such that $f(x) \in {}^\omega x$ for all $x \in X$. Define a regressive function $f_n: X \to \lambda$ by $f_n(x) = f(x)(n)$ for each $n \in \omega$. $W_n = \{f_n^{-1}(\{\gamma\})\colon \gamma <$

$\lambda\} \cap I^+$ is a disjoint I-partition of X and $A_n = \{\gamma < \lambda\colon f_n^{-1}(\{\gamma\}) \in W_n\}$ has the cardinality $\leqslant \eta$. Let $S = \{x \in X\colon x \in \bigcup W_n$ for all $n \in \omega\}$. $S \in I^+$ since $\bigcup W_n \in (I|X)^*$ and I is κ-complete. For any $x \in S$, $f(x) \in \{h : \omega \to \lambda\colon h(n) \in A_n$ for all $n \in \omega\} = B$. Hence $f''S \subset B$ and $|B| \leqslant \eta^\omega < \kappa$. Thus we can find a $g \in B$ such that $\{x \in S\colon f(x) = g\} \in I^+$. I is (ω, λ)-distributive by 1.3. □

The following is straightforward by the remark that follows Lemma 1.3 and by the proof of Lemma 2.6.

Corollary 2.7. (1) *If I is normal η-saturated and $\eta^\mu < \kappa$, then I is (μ, λ)-distributive.*
(2) *If κ is inaccessible and I is normal κ-saturated, then I is strongly normal.*

Now we get a natural generalization of Solovay's theorem.

Theorem 2.8. *Suppose that λ is regular.*
(1) *If I is strongly normal and λ-saturated, then $Sup|X$ is one-to-one for some $X \in I^*$.*
(2) *If I is normal κ-saturated and κ is inaccessible, then $Sup|X$ is one-to-one for some $X \in I^*$.*

Note that κ is assumed to be Mahlo in (1) by Theorem 1.1-(3) and the fact in [12] that no ideal on $\mathcal{P}_{\nu^+}\lambda$ is λ-saturated. The following problem still remains open.

Question 2.9. *Does the conclusion of* 2.8 *also hold for normal λ-saturated ideals, or at least normal κ-saturated ideals?*

On the other hand we can not prove Solovay's theorem for $NS_{\kappa\lambda}$ in ZFC. Baumgartner proved in [4] that $|C| \geqslant \lambda^\omega$ for every closed unbounded $C \subset \mathcal{P}_\kappa\lambda$. We conclude this section by remarking that the regularity of λ in Theorem 2.8 is necessary.

Definition. $SNS_{\kappa\lambda} = \{X \subset \mathcal{P}_\kappa\lambda\colon X \subset \nabla_{\alpha<\lambda} X_\alpha$ for some $\{X_\alpha\colon \alpha < \lambda\} \subset I_{\kappa\lambda}\}$.

It was shown in [13] that $I_{\kappa\lambda} \subsetneq SNS_{\kappa\lambda} \subsetneq NS_{\kappa\lambda}$.

Proposition 2.10. *Suppose that λ is singular and $SNS_{\kappa\lambda} \subset I$. Then for any $X \in I^+$ and $\eta < \lambda$, $Sup|X$ is not $\leqslant \eta$-to-one, that is,*

$$|\{x \in X\colon \sup(x) = \gamma\}| > \eta \quad \textit{for some } \gamma < \lambda.$$

Proof. If $\mathrm{cf}(\lambda) < \kappa$, $\{x\colon \sup(x) = \lambda\} \in I^*_{\kappa\lambda}$. Assume that $\kappa \leqslant \mathrm{cf}(\lambda)$. Let $A \subset \lambda$ be a closed unbounded set of cardinals such that $|A| = \mathrm{cf}(\lambda)$. We first show that $B = \{x \in \mathcal{P}_\kappa\lambda\colon \sup(x) \in A\} \in I^*$. Since $SNS_{\kappa\lambda} \subset I$, we have $\{x\colon \sup(x) \notin x\} \in I^*$. If $\{x\colon \sup(x \cap A) < \sup(x)\} \in I^+$, then we find $\alpha < \lambda$ so that $\{x\colon \sup(x \cap A) < \alpha\} \in I^+_{\kappa\lambda}$ contradicting to A is cofinal. So, $\{x\colon \sup(x \cap A) = \sup(x)\} \in I^*$ and we are done since A is closed.

Let $X \in I^+$ and $\eta < \lambda$. Then, $|X \cap B| \geqslant \lambda > |A|$ and $\sup(x) \in A$ for any $x \in X \cap B$. Now the conclusion is clear. □

3. The minimal ideals in the Rudin–Keisler ordering

The Rudin–Keisler ordering of filters on ω are naturally generalized to ideals on $\mathcal{P}_\kappa\lambda$. In this section we only mention minimal ideals in the ordering.

Definition. Let I and J be ideals on $\mathcal{P}_\kappa\lambda$.

(1) $f_*(I) = \{X \subset \mathcal{P}_\kappa\lambda\colon\ f^{-1}(X) \in I\}$ for any $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$.

(2) $J \leqslant I$ if $J = f_*(I)$ for some $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$.

(3) I and J are said to be *isomorphic* and we write $I \cong J$ if $J = f_*(I)$ and $f|X$ is one-to-one for some $X \in I^*$.

(4) $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ is *I-fine* if $\{x \in \mathcal{P}_\kappa\lambda\colon\ \alpha \notin f(x)\} \in I$ for any $\alpha < \lambda$.

The motivation of I-fine functions will be seen at once.

Proposition 3.1. *For any $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$, f is I-fine iff $f_*(I)$ is an ideal on $\mathcal{P}_\kappa\lambda$.*

So, I is minimal in the RK-ordering iff $I \cong f_*(I)$ for every I-fine f iff every I-fine function is one-to-one on a set in I^*. Menas [13] proved that any normal ultrafilter on $\mathcal{P}_\kappa\lambda$ is minimal if λ is regular or $\mathrm{cf}(\lambda) < \kappa$. We shall give some ideals which are minimal in the RK-ordering in the following. (We simply say *minimal* denoting minimal in the RK-ordering.)

We also define a function which is useful in considering minimality.

Definition. $S : \mathcal{P}_\kappa\lambda \to \lambda + 1$ is *the least unbounded function for I* if

(i) $(\forall\alpha < \lambda)(\{x\colon\ S(x) \leqslant \alpha\} \in I)$ (S is *unbounded*),

(ii) $(\forall g : \mathcal{P}_\kappa\lambda \to \lambda)(X = \{x\colon\ g(x) < S(x)\} \in I^+ \longrightarrow (\exists\alpha < \lambda)(\{x \in X\colon\ g(x) \leqslant \alpha\} \in I^+))$.

I is *weakly normal* if the Sup-function is the least unbounded function for I.

Note that every ideal does not have the least unbounded function. But it is clear that every normal ideal is weakly normal. The following is essentially due to Menas [13].

Theorem 3.2. *Let I be an ideal on $\mathcal{P}_\kappa\lambda$. If the least unbounded function for I is one-to-one on some $X \in I^*$, then I is minimal.*

Proof. Suppose that S is the least unbounded function for I, $X \in I^*$, $S|X$ is one-to-one and f is I-fine. Define $g : \mathcal{P}_\kappa\lambda \to \lambda$ by

$$g(x) = \begin{cases} S(y) & \text{if } (\exists y \in X)(S(y) < S(x) \wedge f(x) = f(y)), \\ 0 & \text{otherwise.} \end{cases}$$

If $\{x\colon\ g(x) > 0\} \in I^+$, then $\{x\colon\ g(x) < S(x)\} \in I^+$. So, we have $Y = \{x\colon\ g(x) \leqslant \alpha\} \in I^+$ for some $\alpha < \lambda$. $Z = f''Y \subset f''\{y \in X\colon\ S(y) < \alpha\}$ and $|\{y \in X\colon\ S(y) < \alpha\}| \leqslant \alpha$ since $S|X$ is one-to-one. So, $|Z| < \lambda$ contradicting to $Z \in f_*(I)^+ \subset I^+_{\kappa\lambda}$.

Now we have proved that $W = \{x\colon\ g(x) = 0\} \in I^*$. $f|W \cap X$ is clearly one-to-one. $\square$

Now the following is immediate by Theorem 2.8 and 3.2.

Corollary 3.3. *For regular λ, every strongly normal λ-saturated ideal is minimal.*

Before extending this to λ such that $\mathrm{cf}(\lambda) < \kappa$, we state results from [3].

Definition. Suppose that κ is inaccessible and $\{s_\alpha\colon\ \alpha < \lambda^{<\kappa}\}$ is an enumeration of $\mathcal{P}_\kappa\lambda$. If $g : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda^{<\kappa}$ is defined by

$$g(x) = \{\alpha < \lambda^{<\kappa}\colon\ s_\alpha \subset x \text{ and } |s_\alpha| < |x \cap \kappa|\},$$

g is one-to-one and $g_*(I)$ is an ideal on $\mathcal{P}_\kappa\lambda^{<\kappa}$. Moreover,

Lemma 3.4. *$g_*(I)$ is strongly normal iff I is strongly normal. Also for any η, $g_*(I)$ is η-saturated iff I is η-saturated.*

Theorem 3.5. *If κ is Mahlo and $\mathrm{cf}(\lambda) < \kappa$, then every strongly normal λ^+-saturated ideal is minimal.*

Proof. $\lambda^{<\kappa} = \lambda^+$ and $g_*(I)$ is strongly normal λ^+-saturated ideal on $\mathcal{P}_\kappa\lambda^+$ by 1.2 and 3.4. Hence there is an $X \in g_*(I)^*$ such that $Sup|X$ is one-to-one.

Suppose that $h : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ is I-fine. Define $k : \mathcal{P}_\kappa\lambda^+ \to \mathcal{P}_\kappa\lambda$ so that $k(g(x)) = h(x)$ for any $x \in \mathcal{P}_\kappa\lambda$. Then $h_*(I) = k_*(g_*(I))$. By the argument used in Theorem 3.2, $k|Y$ is one-to-one for some $Y \in g_*(I)^*$. Hence $h|g^{-1}(Y)$ is injective and $g^{-1}(Y) \in I^*$. □

In [2] we proved the converse of Theorem 3.2 for prime ideals provided that λ is regular. Here is a more general version.

Theorem 3.6. *Suppose that λ is regular, I has the least unbounded function S and there is no disjoint collection of λ many I-positive sets. Then, I is minimal iff $S|X$ is one-to-one for some $X \in I^*$.*

Proof. One direction was proved in Theorem 3.2. So, assume that I is minimal. We use Solovay's theorem in [14] that every stationary subset of a regular cardinal δ is a disjoint union of δ many stationary subsets. So, let $\{A_\alpha\colon\ \alpha < \lambda\}$ be a pairwise disjoint family of stationary subsets of λ such that $\bigcup_{\alpha<\lambda} A_\alpha = \{\xi < \lambda\colon\ \mathrm{cf}(\xi) = \omega\}$. Define $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ by $f(x) = \{\alpha < \lambda\colon\ A_\alpha \cap S(x)$ is a stationary subset of $S(x)\}$. We first prove that f is I-fine.

Suppose that there is an $\alpha < \lambda$ such that $\{x \in \mathcal{P}_\kappa\lambda\colon\ \alpha \notin f(x)\} \in I^+$. $X = \{x \in \mathcal{P}_\kappa\lambda\colon\ A_\alpha \cap S(x)$ is not stationary in $S(x)\} \in I^+$. For each $x \in X$, there is a closed unbounded subset of $S(x)$, C_x such that $C_x \cap A_\alpha = \emptyset$. Let $D = \{\beta < \lambda\colon\ \{x \in X\colon\ \beta \notin C_x\} \in I\}$.

We show that D is an ω_1-closed unbounded subset of λ. Let $\gamma_0 < \lambda$. For each $x \in \{y \in X\colon\ \gamma_0 < S(x)\} \in I^+$, we have a $g(x) \in C_x$ such that $\gamma_0 \leqslant g(x) < S(x)$. To show $\{x\colon\ g(x) < \gamma_1\} \in (I|X)^*$ for some $\gamma_1 < \lambda$, we make a sequence $\{\xi_\alpha\colon\ \alpha < \eta\}$

with $\eta < \lambda$ such that $Y_\alpha = \{x \in X:\ \xi_\alpha < g(x) \leqslant \xi_{\alpha+1}\} \in I^+$ as follows. Let $\xi_0 = \gamma_0$. Suppose that $\{\xi_\beta:\ \beta < \alpha\}$ was already defined for $\alpha < \lambda$ and $\nu = \sup\{\xi_\beta:\ \beta < \alpha\}$. By regularity of λ, $\nu < \lambda$. If $\{x \in X:\ \nu < g(x)\} \in I$, let $\gamma_1 = \nu$. Otherwise, we have a $\xi_\alpha > \nu$ so that $\{x \in X:\ \nu < g(x) \leqslant \xi_\alpha\} \in I^+$ since S is the least unbounded function for I. We can not continue this argument λ times since $\{Y_\xi:\ \xi < \eta\}$ is a disjoint collection of I-positive sets.

Now $\{x:\ C_x \cap [\gamma_0, \gamma_1] \neq \emptyset\} \in (I|X)^*$ with $\gamma_1 = \sup\{\xi_\alpha:\ \alpha < \eta\} < \lambda$. We continue this procedure to get an increasing sequence $\{\gamma_n:\ n \in \omega\}$ such that $X_n = \{x:\ C_x \cap [\gamma_n, \gamma_{n+1}] \neq \emptyset\} \in (I|X)^*$ for each $n \in \omega$. Let $Z = \bigcap\{X_n:\ \in \omega\} \in (I|X)^*$ and $\delta = \sup\{\gamma_n:\ n \in \omega\}$. $\delta < \lambda$ and $\delta \in C_x$ for any $x \in Z$ by our construction. So, $\gamma \leqslant \delta \in D$ and D is unbounded.

For the ω_1-closure of D, let $\{\beta_n:\ n \in \omega\} \subset D$ be any increasing sequence and $\beta = \sup\{\beta_n:\ n \in \omega\}$. Since $Z_n = \{x:\ \beta_n \in C_x\} \in (I|X)^*$ for each $n \in \omega$, $Z = \bigcap_{n\in\omega} Z_n \in (I|X)^*$. For each $x \in Z$, we have $\{\beta_n:\ n \in \omega\}$ is an increasing sequence in C_x which is closed. Hence $\beta \in C_x$ for any $x \in Z$. Thus $\beta \in D$.

Since we have proved D is ω_1-closed unbounded, there is a $\beta \in A_\alpha \cap D$. $\{x:\ \beta \in C_x\} \in (I|X)^*$ since $\beta \in D$. But $C_x \cap A_\alpha = \emptyset$ for any $x \in X$. This contradiction tells us that f is I-fine.

We can find an $X \in I^*$ such that $f|X$ is one-to-one since I is minimal. It is clear that $f(x) = f(y)$ if $S(x) = S(y)$. So, $S|X$ is one-to-one. □

As the referee kindly pointed out, the original paper had contained an error to assert the conclusion also holds if $\bigcap_{n\in\omega} X_n \in I^+$ for any decreasing sequence of I-positive sets $\{X_n:\ n \in \omega\}$. In this direction one may need a slightly stronger condition than (ω, λ)-distributivity as below.

For any $X \in I^+$ and $\{W_n:\ n \in \omega\}$ such that $W_0 = \{X\}$, W_{n+1} consists of disjoint I-partitions of each element of $\bigcup W_n$ and $|W_n| \leqslant \lambda$ for all $n \in \omega$, there are $Y \in I^+$ and $\{X_n:\ n \in \omega\}$ such that $X_n \in \bigcup W_n$ and $Y - X_n \in I|X$ for every $n \in \omega$.

Note that the existence of an ideal which satisfies the condition of Theorem 3.6 implies a weakly normal ideal in the sense of [1]. The theorem also tells us for regular λ a normal λ-saturated ideal is minimal iff $Sup|X$ is one-to-one for some $X \in I^*$.

The author does not know whether $NS_{\kappa\lambda}$ is minimal. We can easily find a function $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$ such that $f^{-1}(\{a\}) \in I_{\kappa\lambda}$ for all $a \in \mathcal{P}_\kappa\lambda$ and $f|X$ is not one-to-one for any $X \in I^+_{\kappa\lambda}$. But no I-fine such a function could be found by the author.

It is a well-known theorem for ideals on κ that normal ideals are minimal. Also, any distinct ultrafilters on κ extending the closed unbounded filter are not isomorphic. These are not the case for ideals on $\mathcal{P}_\kappa\lambda$. Two distinct isomorphic fine measures extending the closed unbounded filter are presented for λ strong limit in [13] and [2]. The comparative simplicity in the theory of ideals on κ seems to come from the following trivial facts.

1. If $|X| < \kappa$, then $|f''X| < \kappa$ for any $f : \kappa \to \kappa$.
2. If $|Y| < \kappa$ and $f : \kappa \to \kappa$ is one-to-one, then $|f^{-1}(Y)| < \kappa$

The following is of some interest in this sense.

Definition. $\nabla J = \{X \subset \mathcal{P}_\kappa\lambda$: there is a regressive $f : X \to \lambda$ such that $f^{-1}(\{\alpha\}) \in J$ for any $\alpha < \lambda\}$.

So, J is normal iff $\nabla J = J$. It was shown in [6] that $SNS_{\kappa\lambda} = \nabla I_{\kappa\lambda}$ and $NS_{\kappa\lambda} = \nabla SNS_{\kappa\lambda}$.

Proposition 3.7. *Let I, J be ideals on $\mathcal{P}_\kappa\lambda$. If $I \cong f_*(I)$ and $\nabla J \subset I \cap f_*(I)$, then one of the following holds.*

(a) *There is $X \in I_{\kappa\lambda}$ such that $f^{-1}(X) \in J^+$.*

(b) *There is $Y \in J^+$ such that $f^{-1}(Y) \in I_{\kappa\lambda}$.*

If I is normal, then (b) is satisfied.

Proof. Let $Z \in I^*$ such that $f|Z$ is one-to-one, $Z_0 = \{x \in Z\colon x \not\subset f(x)\}$ and $Z_1 = \{x \in Z\colon f(x) \not\subset x\}$. Either Z_0 or Z_1 is in I^+.

Assume $Z_0 \in I^+$. For any $x \in Z_0$, there is $\alpha_x \in x$ such that $\alpha_x \notin f(x)$. By our hypothesis $Z_0 \in (\nabla J)^+$. Hence for some γ, $A = \{x \in Z_0\colon \gamma \notin f(x)\} \in J^+$. Then $X = f''A \subset \{y\colon \gamma \notin y\} \in I_{\kappa\lambda}$.

If $Z_1 \in I^+$, let $B = f''Z_1 \in f_*(I)^+$. For each $y \in B$, we have $\beta_y \in y$ with $\beta_y \notin f^{-1}(y)$. Since $B \in (\nabla J)^+$, there is δ such that $Y = \{y \in B\colon \delta \notin f^{-1}(y)\} \in J^+$. Then, $f^{-1}(Y) \subset \{x\colon \delta \notin x\} \in I_{\kappa\lambda}$.

We can easily see that $\{x\colon x \subset f(x)\} \in I^*$ for any normal ideal I and I-fine f. Let $X_\alpha = \{x\colon \alpha \notin f(x)\} \in I$ for each $\alpha < \lambda$. $\{x\colon x \not\subset f(x)\} \subset \nabla_{\alpha<\lambda} X_\alpha \in I$ by normality. □

This proposition says that, in some sense, a "wild" function is necessary for large ideals to be isomorphic.

We can consider the properties corresponding to P-point and Q-point for ideals on $\mathcal{P}_\kappa\lambda$. Although some interesting results were achieved in [16] and [8], the structural theory of ideals on $\mathcal{P}_\kappa\lambda$ is not satisfactorily established yet. $\mathcal{P}_\kappa\lambda$ seems much richer than κ.

References

[1] Y. Abe, Weakly normal ideals on $\mathcal{P}_\kappa\lambda$ and the singular cardinal hypothesis, Fund. Math. 143 (1993) 97–106.

[2] Y. Abe, Weakly normal filters and the closed unbounded filter on $\mathcal{P}_\kappa\lambda$, Proc. Amer. Math. Soc. 104 (1988) 1226–1234.

[3] Y. Abe, Saturation of fundamental ideals on $\mathcal{P}_\kappa\lambda$, J. Math. Soc. Japan, to appear.

[4] J.E. Baumgartner, On the size of closed unbounded sets, Ann. Pure Appl. Logic 54 (1991) 195–227.

[5] J.E. Baumgartner, A.D. Taylor and S. Wagon, Structural properties of ideals, Dissertationes Math. CXCVII (1982).

[6] D.M. Carr, The minimal normal filter on $\mathcal{P}_\kappa\lambda$, Proc. Amer. Math. Soc. 86 (1982) 316–320.

[7] D.M. Carr, J.P. Levinski and D.H. Pelletier, On the existence of strongly normal ideals on $\mathcal{P}_\kappa\lambda$, Arch. Math. Logic 30 (1990) 59–72.
[8] D.M. Carr and D.H. Pelletier, Towards a structure theory for ideals on $\mathcal{P}_\kappa\lambda$, in: Set Theory and its Applications, J. Steprāns and S. Watson, eds., Lecture Notes in Math. 1401 (Springer, Berlin, 1989) 41–54.
[9] M. Foreman, Potent axioms, Trans. Amer. Math. Soc. 294 (1986) 1–28.
[10] T.J. Jech, Some combinatorial problems concerning uncountable cardinals, Ann. Math. Logic 5 (1973) 165–198.
[11] C.A. Johnson, Some partition relations for ideals on $\mathcal{P}_\kappa\lambda$, Acta Math. Hung. 56 (1990) 269–282.
[12] Y. Matsubara, Menas' conjecture and generic ultrapowers, Ann. Pure Appl. Logic 36 (1987) 225–234.
[13] T.K. Menas, On strong compactness and supercompactness, Ann. Math. Logic 7 (1974) 327–359.
[14] R.M. Solovay, Real valued measurable cardinals, in: Axiomatic Set Theory, Proc. Sympos. Pure Math. 13, I (Amer. Math. Soc., Providence, RI, 1971) 397–428.
[15] R.M. Solovay, Strongly compact cardinals and the GCH, in: Axiomatic Set Theory, Proc. Sympos. Pure Math. 25 (Amer. Math. Soc., Providence, RI, 1974) 365–372.
[16] W. Zwicker, A beginning for structural properties of ideals on $\mathcal{P}_\kappa\lambda$, in: Set Theory and its Applications, J. Steprāns and S. Watson, eds., Lecture Notes in Math. 1401 (Springer, Berlin, 1989) 201–217.